Analytical Benthic Flux Model Forced by Surface-Water Waves: Application to the South Atlantic Bight, USA

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ABSTRACT
Benthic flux is the flow rate per unit area of some physical, chemical, or biological property across the sediment-water interface (bed) of a water body. Riedl et al. (1972) used Reid and Kajiura's (1957) classic solution for the damping of a linear, surface-water wave by an infinitely deep, rigid, porous medium, to develop an analytical expression for benthic water flux forced by surface-water waves. The objectives of this work are to: (1) re-cast the Riedl et al. (1972) solution to observe Reid and Kajiura's (1957) use of a complex wave number; (2) detail an example calculation using Moore's (1996) South Atlantic Bight (SAB) study area; and (3) suggest that benthic discharge flux forced by surface-water waves explains Moore's (1996) observation of excess ^{226}Ra on the inner shelf of the SAB.

INTRODUCTION
Benthic flux (q_{bf}) is a vector quantity, oriented normal to a sediment-water interface. A benthic recharge flux (q_{br}) vector is oriented from a surface-water body toward the underlying porous medium; a benthic discharge flux (q_{bd}) vector is oriented from the porous medium toward the surface-water body. The units of q_{bf} depend on the flux property; for example, [$L^3T^{-1}L^{-2}=LT^{-1}$] for a benthic volume flux or [$MT^{-1}L^{-2}$] for a benthic mass flux, where L, M, and T are length, mass, and time dimensions, respectively. Various potential gradients across the sediment-water interface force q_{bf}. For example, a concentration gradient forces a Fickian diffusive benthic flux, and a pressure gradient forces an advective benthic flux. Benthic flux occurs in palustrine, lacustrine, riverine, estuarine, and marine environments. Related terms exist in the literature; for example, submarine ground-water discharge is a benthic discharge flux of water ($q_{bd.w}$) to an overlying marine water body, and the sub-tidal pump is a benthic flux of water ($q_{bf.w}$) forced by surface-water waves. Geographically constrained variations in porous-medium and surface-water characteristics govern q_{bf}. For example, Moore and Wilson (2005) explain that tidal pumping, episodic storm events, and leakage from underlying geologic units mix surface and pore water in the South Atlantic Bight (SAB). Precht et al. (2004) describe sediment oxygen dynamics forced by the interaction of wave-generated currents on a rippled flume bed.

FORMULATION
Reid and Kajiura (1957) solved a two-dimensional, x-z oriented, boundary value problem in which a linear surface-water wave of deep-water amplitude a_0 and period T, propagates through a domain described by Cartesian space dimensions x and z, where z is parallel to the gravity (g) vector. The surface-water wave generates pressure gradients in both the water column and the underlying rigid, porous, isotropic medium of permeability k. The water has density ρ, dynamic viscosity μ, and kinematic viscosity ν. The mean water surface is located at z=0; the sediment-water interface at z=-h; wave inception is in deep water, at x=0; the displacement η of the water surface in response to the wave is described by $\eta = a_0 e^{i(\lambda x - \sigma t)}$, where $\sigma=2\pi/T$ is the angular frequency of the wave, $\lambda=\lambda_r+i\lambda_i$ is a complex wave number with real λ_r and imaginary λ_i components, $i=\sqrt{-1}$, and t is time. The governing equations require that the Laplacian ∇^2 of the velocity potential $\phi(x,z,t)$ in the surface-water domain and pressure $p(x,z,t)$ in the ground-water domain equal zero ($\nabla^2\phi = 0$ and $\nabla^2 p = 0$). Reid and Kajiura (1957) employ four

boundary conditions: a dynamic free-surface boundary condition $g\eta = \partial\phi/\partial t$ at z=0; a kinematic free-surface boundary condition $w = \partial\phi/\partial t$ at z=0, where w is the vertical velocity component; a dynamic interface boundary condition p_s=p at z=-h, where subscript $_s$ denotes the value in the porous domain, and an absence of the subscript denotes the value in the surface domain; and a kinematic interface boundary condition w_s=w at z=-h. Reid and Kajiura (1957) assume solutions of the following form:

$$\phi = [A\cosh(\lambda h + \lambda z) + B\sinh(\lambda h + \lambda z)]\, e^{i(\lambda x - \sigma t)} \quad (1)$$

$$p_s = Ce^{\lambda h + \lambda z} e^{i(\lambda x - \sigma t)} \quad (2)$$

where A, B, and C are unknowns, and w_s = 0 at $z = -\infty$. Reid and Kajiura (1957) use the following relationships at $z = -h$: the Bernoulli Equation $p = \rho\,\partial\phi/\partial t$, the definition of velocity potential $w = -\partial\phi/\partial z$, and Darcy's Law $q_{bf.w} = w_s = -(k/\mu)\partial p_s/\partial z$, with the above-described boundary conditions to solve for A, B, C and λ. It can then be shown (King, 2007) that

$$\boxed{q_{bf.w} \approx -\hat{A}[\cos(\lambda_r x - \sigma t) - \beta\sin(\lambda_r x - \sigma t)]} \quad (3)$$

where $\hat{A}$ is the amplitude of $q_{bf.w}$, β is a $q_{bf.w}$ amplification parameter,

$$\hat{A} = kg\lambda_r a_0 e^{-\lambda_i x} / [\nu\cosh(\lambda_r h)] \quad (4)$$

$$\sigma^2 \approx g\lambda_r \tanh(\lambda_r h) \quad (5)$$

$$\lambda_i \approx 2R\lambda_r / [2\lambda_r h + \sinh(2\lambda_r h)] \quad (6)$$

$$R = \sigma k/\nu \quad (7)$$

$$\beta = \tanh(\lambda_r h)(R - \lambda_i h) + \lambda_i/\lambda_r \quad (8)$$

This formulation of Reid and Kajiura's (1957) solution includes wave damping. The Riedl et al. (1972) formulation does not include wave damping, but does include a bed slope term. Equation 4 is equivalent to the Riedl et al. (1972) amplitude, where the decayed, deep-water, wave amplitude $a_0 e^{-\lambda_i x}$ in the current formulation is replaced by a local wave amplitude a, such that $a = a_0 e^{-\lambda_i x}$. Benthic water flux in both formulations integrates to zero over one wave period, such that $\int_t^{t+T} q_{bf.w}\, dt = 0$. Gross $q_{bd.w}$ at any point, averaged over the wave period ($\overline{q}_{bd.w}$), is

$$\overline{q}_{bd.w} \approx (1/\sigma T)\int_{t+\sigma T/4}^{t+3\sigma T/4} q_{bf.w}\, dt = \hat{A}/\pi \quad (9)$$

where β is small (β<0.1) and $\eta/a = 1$ at $\lambda_r x - \sigma t = 0$. The current formulation utilizes the wave-generated pressure gradient across a planar bed to yield Equation 3. This pressure gradient is a function of the permeability of the porous medium that constitutes the planar bed, but not a function of bed roughness. Investigations—such as Precht et al. (2004)—of wave-generated currents interacting with a rippled bed address a related but different process, in which q_{bf} is a function of bed roughness.

APPLICATION

Riedl et al. (1972) detail parameters necessary to apply Equation 9 to Moore's (1996) study area (the inner shelf of the SAB, from shore to 20*km* offshore, over a 320-*km*-long section of coast, from the Savannah River to Cape Fear). Specifically, $0 < h < 18m$; $9.9\times10^{-12}m^2 < k < 2.0\times10^{-11}m^2$; and wave spectra, which show that $T < 6s$ more than 50% of the time. Assuming $k=10^{-11}m^2$, $\nu=1.17\times10^{-6}m^2/s$, $T=5.5s$, and $a_0=0.5m$ at $X=20km$ in Moore's (1996) study area, where X is a shore-normal distance ($X=0m$ at shore), then $\overline{q}_{bd.w}$ integrated over the study area ($\overline{Q}_{bd.w}$) is

$$\overline{Q}_{bd.w} = \iint \overline{q}_{bd.w} dydX = \int_{10m}^{20km}\int_{0}^{320km} \frac{\hat{A}}{\pi} dydX \approx 8{,}100\, m^3/s \qquad (10)$$

where y is shore parallel. Bathymetric contours are assumed to be locally straight and parallel to the coast, and the wave field oriented such that it does not refract. A well-known wave breaking condition is assumed, where $2a/h > 0.78$. The wave shoals as a function of a_0, deep-water wave speed, and wave-group velocity; the wave damps as a function of λ_i and x; and $\beta<10^{-5}$ is small. Maximum $\hat{A}=1.3\times10^{-5}m/s$ occurs at $X=37m$, where the shoaled, damped wave breaks. Shore-proximate $\hat{A}=7.9\times10^{-6}m/s$ at $X=10m$; deep-water $\hat{A}=1.0\times10^{-6}m/s$ at $X=20km$; 99% of $\overline{Q}_{bd.w}$ is generated offshore of the break point at $X=37m$. If $T=6.0s$ and $a_0=0.55m$, then $\overline{Q}_{bd.w} \approx 9{,}500\, m^3/s$; if $T=5.0s$ and $a_0=0.45m$, then $\overline{Q}_{bd.w} \approx 6{,}600\, m^3/s$.

DISCUSSION AND CONCLUSIONS

Moore (1996) observed 0.19*dpm*/ℓ average ^{226}Ra activity for inner-shelf surface waters, and suggested maximum contributions of 0.01 and 0.08*dpm*/ℓ to the inner shelf from estuaries and the ocean, respectively. He determined that 2.1×10^{11}*dpm*/*d* excess ^{226}Ra must then be explained by some other source, by assuming an inner-shelf volume of 6.4×10^{13}ℓ and a 30-day residence time for the unexplained 0.10*dpm*/ℓ [from $(0.19 - 0.01 - 0.08)(6.4\times10^{13}) / 30 = 2.1\times10^{11}$]. Moore (1996) then assumed an inner-shelf, pore-water ^{226}Ra activity equivalent to an observed 7*dpm*/ℓ ^{226}Ra activity for brackish ground water at North Inlet, South Carolina, to suggest that $Q_{bd.w} = Q_{Moore} = 3\times10^{10} l/d = 350 m^3/s$ reasonably generates the 2.1×10^{11}*dpm*/*d* excess ^{226}Ra $\left(2.1\times10^{11}/7 = 3\times10^{10}\right)$. Younger (1996) invoked a mass balance argument to suggest that benthic freshwater discharge to Moore's (1996) study area is approximately $14m^3/s$ (4% of Q_{Moore}). Li et al. (1999) identified two additional near-shore, physical processes capable of forcing $\overline{Q}_{bd.w}$: near-shore tidal pumping generates $\overline{Q}_{bd.w} = 130\, m^3/s$ (37% of Q_{Moore}), and wave set-up generates $\overline{Q}_{bd.w} = 190\, m^3/s$ (54% of Q_{Moore}). Li et al. (1999) suggest that $\overline{Q}_{bd.w}$ forced by wave set-up, tidal oscillation, and Younger's (1996) benthic freshwater discharge can be summed linearly, such that a composite $\overline{Q}_{bd.w}$ explains 95% of Q_{Moore}.

Moore and Wilson (2005) measured 1.3, 2.0, 3.5, and 8.0*dpm*/ℓ pore water ^{226}Ra activity at 0.50, 1.00, 1.25, and 1.75*m* depths below the bed, on the outer edge of Moore's (1996) study area. Surface-water ^{226}Ra activity was 0.2*dpm*/ℓ. Based on these observations, Moore's (1996) 7*dpm*/ℓ ^{226}Ra activity assumption may be high for inner-shelf pore water just below the bed. Assume 0.5*dpm*/ℓ ^{226}Ra activity for inner-shelf pore water just below the bed and recognize that $\overline{Q}_{bd.w} = \overline{Q}_{br.w}$. Moore's (1996) implicit assumption that inner-shelf ^{226}Ra lost to q_{br} is small compared to inner-shelf ^{226}Ra gained by q_{bd} is not appropriate in the current formulation, where surface-water and pore-water ^{226}Ra activity are of the same order of magnitude. Then, $\overline{Q}_{bd.w} = 2.1\times10^{11} dpm/d \div (0.5 dpm/\ell - 0.2 dpm/\ell) = 7\times10^{11}\ \ell/\text{d} = 8{,}100 m^3/s$. The $0.5 dpm/\ell$ ^{226}Ra activity is reasonable in that it agrees with Moore and Wilson (2005), is greater than surface-water activity but less than the activity in deeper portions of the sediment column. The linearly summed trio of near-shore forcing mechanisms examined by Li et al. (1999) and Younger (1996) then explain 334/8100=4% of the updated excess ^{226}Ra activity. The higher-magnitude $\overline{Q}_{bd.w} = 8{,}100 m^3/s$ forced by surface-water waves over the entire study area, detailed in Equation 10, with an updated pore water ^{226}Ra activity assumption, explains Moore's (1996) excess ^{226}Ra observation.

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