Submarine groundwater discharge as an inverse problem

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ABSTRACT

In this work we present a new mathematical method that allows the recovering of lacking data for an aquifer domain having over-specified boundary data on a part of its boundary.

We will focus, in the framework of this work, on applying this method to quantify the submarine groundwater discharge amounts. We present in this paper two synthetic examples of implementation of the method and a real case of a deep coastal sheet of the south of Tunisia.

INTRODUCTION

The major part of the world's usable freshwater lies underground and wherever, aquifers are hydraulically connected to the ocean, submarine groundwater can discharge into the sea and salt water can intrude landward into freshwater. Or, for many years, coastal groundwater studies were focused on the latter phenomena because of its direct consequences for water supplies. Little attention was paid to submarine groundwater discharge (SGD), because it was considered to be insignificant compared to the discharge from rivers and other surface waters. However, in recent years, SGD has received more attention because it appears that is more than a simple exchange of water between land and sea. In fact, the flow of groundwater into ocean is critical because this water is often carried with dissolved nutrients and pollutants. So it is, now, recognized that SGD can influence coastal-water and geochemical budgets and drive ecosystem change.

Estimation of the magnitude of SGD is a real challenge, because of the difficulty to localize the SGD area and to distinguish between the fresh groundwater discharge and the discharge of the seawater re-circulation. As presented in (Burnett *et al*. 2006) different methods are used to measure SGD. We can cite direct or indirect field measurement like seepage meters, piezometers or the use of geochemical tracers such as radium isotopes and radon.The application of these methods involves considerable financial and effort investments. So that scientists resort to numerical methods, like the water balance approach or computational simulations.

Numerical models and software used in SGD simulations are of different complexity but all of them have in common the resolution of a forward well-posed problem (Bear *et al*. 1999).

Here we consider the problem of estimating SGD quantities as an **inverse problem**, in which the interface between the landward and the sea is a lacking data boundary. We will determine the hydraulic head and flux on this boundary from the knowledge of over-specified data (hydraulic head and flux) on the other boundary of the domain. It's a data completion problem. To our knowledge it's the first time that the SGD is estimates as an inverse problem.

MATHEMATICAL PROBLEM

The model

We will focus, here, on extending the data completion algorithm based on an energy error functional, initially introduced for the Laplace equation (Andrieux *et al*. 2006) to the Darcy framework (Escriva *et al*. 2007, Hariga *et al*. 2008). The actual application concerns the identification of submarine groundwater discharge (SGD) with known transmissivities and prescribed piezometric levels on a part of the domain boundary, in steady state conditions.
Using the notations shown on Fig. 1, a simplified mathematical model is given by :

$$\begin{cases} div(-T\nabla h) = S & in \quad \Omega \\ h = H & on \quad \Gamma_m \\ -T\dfrac{\partial h}{\partial n} = \Phi & on \quad \Gamma_m \end{cases} \qquad (1)$$

Where *T* is the transmissivity, the hydraulic head is denoted by *h*, *S* is the source term. Γ_m, is the portion of the boundary of Ω where both *h* and its normal derivative are known (i.e. an over-specified condition) and we denote by Γ_u, the part of the boundary where all information is lacking.

This situation can correspond to an arbitrary boundary for an aquifer known to extend over Γ_u but where no information is available. It is for example the case of a deep aquifer continuing beyond the shoreline below the sea bottom. Indeed from a practical viewpoint, as the known piezometric levels correspond to measured values in boreholes and are presented as interpolated isolines, it is easy to calculate their normal derivative even on an internal part of the domain but close to the boundary isopiezometric line, in order to obtain the over-specified data.

We propose, in this paper, to reconstruct the missing data using an energy-like error functional introduced in (Andrieux *et al*. 2006) and then to determine according to the sign of the flux the submarine groundwater discharge or recharge values.

The Data Completion Problem

Let us consider the above Cauchy problem (1) with over-specified data on Γ_m. It is known since Hadamard (Hadamard 1953) to be ill-posed in the sense that the dependence of the solution of the Partial Differential Equation and consequently of the lacking boundary data is not continuous. Provided the data *H* are compatible with the flux Φ, solving the Cauchy problem can be state as follows: solving problem (1) and finding (H_u, Φ_u) respectively the hydraulic head and the flux on Γ_u.

The approach in the energy-like error functional method developed in (Andrieux *et al*. 2006) follows in two steps. First, we consider, for a given pair (η, τ), the two following mixed well-posed problems (2-a & 2-b):

$$\text{a)} \begin{cases} div(-T\nabla h_1) = S & in \quad \Omega \\ h_1 = H & on \quad \Gamma_m \\ -T\dfrac{\partial h_1}{\partial n} = \eta & on \quad \Gamma_u \end{cases} \qquad \text{b)} \begin{cases} div(-T\nabla h_2) = S & in \quad \Omega \\ h_2 = \tau & on \quad \Gamma_u \\ -T\dfrac{\partial h_2}{\partial n} = \Phi & on \quad \Gamma_m \end{cases} \qquad (2)$$

The second step is to build an energy-like error functional on the pair (η, τ), using an energy norm, denoted E:

$$E(\eta,\tau)= \frac{1}{2}\int_{\Omega} T\,\nabla(h_1-h_2) \quad \nabla(h_1-h_2)d\Omega \tag{3}$$

Using the proprieties of h_1 and h_2 as defined in (3), we easily derive an alternate expression of E:

$$E(\eta,\tau)= \int_{\Gamma_u} T(\eta-\nabla(h_2.n))(h_1-\tau) \quad d\Gamma + \int_{\Gamma_m} T(\nabla(h_1.n)-\Phi)(H-h_2) \quad d\Gamma \tag{4}$$

Indeed, the functional E vanishes when the pair (η, τ) meets the real data (ϕ_u, H_u) on the boundary Γ_u. Then $h_1 = h_2 + cst$ and the Cauchy problem (1) is solved. For more details see Andrieux *et al*.

To conclude, the leaking data are obtained via the following minimization algorithm (5):

$$(\Phi_u, H_u)= \quad \arg\min \quad E(h_1(\eta,H), h_2(\Phi,\tau)) \tag{5}$$

with h_1 and h_2 solutions of (2) and $(\eta,\tau) \in H^{-1/2}(\Gamma_u) \times H^{1/2}(\Gamma_u)$

NUMERICAL TESTS

The goal of the numerical experiment is to evaluate the volumic flow rate Q_v ($Q_v = \int_{\Gamma_u} T\frac{\partial h}{\partial n} \quad ds$) over the inaccessible interface between the sea and the land, by exploiting over-specified measurements on an other part of the domain's boundary.

The data completion methodology consists in solving the Cauchy problem (1) where the over-specified data (H,Φ) are extracted from the direct problem.

We present two cases: the first with intrusion water from the sea to the land and the second with discharge water from the land to the sea. The two cases are studied with the same geometry: a square of 1000mx1000m with three heterogeneous zones (*T_1 =20 m^2/day, T_2 =10 m^2/day* and *T_3 =5 m^2/day*) and three pumping wells (*Q_1 =-4 m^3/day, Q_2 =-10 m^3/day* and *Q_3 =-3 m^3/day*). Vertical boundaries are impervious, the horizontal lower boundary is the over-specified one (Γ_m) and the horizontal upper is the lacked boundary (Γ_u) and it represents the sea.

The domain is meshed with a regular mesh of triangular elements with linear interpolation, characterized by 1038 nodes and 495 elements.

Case of dicharge:

In this case we impose a constant Neuman condition (*Φ= -0.1m^2/day*) over Γ_m and as it is shown on Figure 1 the flow of water is to the sea (Γ_u) from the land. Figure 2 shows the reconstruction data (the hydraulic head and its normal derivative) over the boundary where data are missing, the upper one; and we can denote that far from the corner of the square the reconstruction is quite satisfying.

In this case *Q_v^{exact} = 100 m^3/day* and *$Q_v^{compute}$ = 98.6 m^3/day*, so the error is **1.4%.**

Case of intrusion:

In this case we impose a constant hydraulic head (*H = 10m*) over Γ_m .

In this case *Q_v^{exact} = -105.44 m^3/day* and *$Q_v^{compute}$ = 105.33 m^3/day*, so the error is less than **1%.**

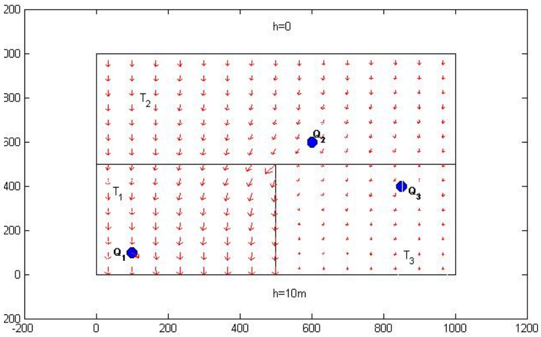

Figure 1. Arrows of (T δh/δx, T δh/δy) in the case of water discharge

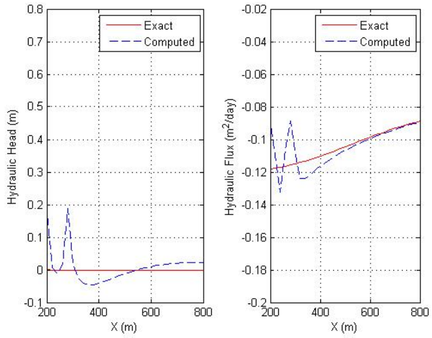

Figure 2. Hydraulic head and its normal derivative over Γ_u (discharge case).

THE CASE OF DJEFFARA MIOCENE COASTAL AQUIFER:

The implementation of this method on the steady state of the deep aquifer of Djeffara, South East of Tunisia, relates on the determination of the location of the limit of zero head under the Mediterranean sea and to the flows exchanged between the aquifer and the sea on this level.

CONCLUSIONS

In this paper the sea water discharge/recharge is evaluated as a data completion problem by solving a Cauchy problem derived from Darcy's equations.

The Cauchy problem is solved by minimizing an energy-like functional which depends on the hydraulic head and flux over the interface between the sea and the land.

At our knowledge it is the first time that the problem of SWD is considered as an inverse problem and the present results are very satisfying and encouraging.
We are working on applying this method to a real case.

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